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Algebraic Principles of Precise Point GNSS Positioning with CDMA Signals

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Abstract. The paper deals with the algebraic principles of Precise Point GNSS Positioning based on CDMA signals. The client-side integer ambiguity resolution algorithms are explained within the framework of the rigorous mathematical theory, and their performance is discussed. Significant reduction in convergence times required to achieve user positioning accuracy of about 1cm is reported. In our future publications a similar approach will be adopted to tackle the network solution.

Keywords: satellite navigation, space vehicle (SV), Precise Point Positioning (PPP), ambiguity resolution (AR), Float PPP, Integer PPP

Introduction

Precise Point Positioning (PPP) [1] is used to calculate absolute (i.e., relative to the Earth-centered reference frame) coordinates of user receivers with errors not exceeding 1 cm by means of joint processing of code and carrier phase measurements with additional use of precise satellite corrections. Two PPP processing methods can be distinguished: (a) Integer PPP which includes the ambiguity resolution of phase measurements and (b) Float PPP without such resolution. In what follows 'a user solution' is a computational procedure employed by users to calculate the coordinates of their receivers (antenna phase centers, to be precise) by either Float or Integer PPP method with the use of precise satellite corrections. 'A network solution', by way of contrast, stands for a set of computational algorithms, running (in the case of an industrial setup) on the network server and continuously estimating above-mentioned precise satellite corrections by common processing of code and carrier phase measurements generated by the network of ground stations.

The main issue of Integer PPP processing is in overcoming the rank deficiency problem of the system of linear equations obtained by the linearization of generally nonlinear mathematical models for code and phase measurements. The algebraic principles of Float PPP method are currently well understood. In particular, it has been shown [1] that rank deficiency can be tackled by taking into account code and carrier phase biases and introducing them in the said mathematical models by combining with clocks and carrier phase ambiguities. The number of unknowns in such a system of linear equations thus drops down to the value of its rank. This allows for unambiguous computation of user coordinates as well as of the values of 'lumped' parameters which appear as a result of the above-mentioned combining procedure. However, such a procedure entails the loss of the integer nature of carrier phase ambiguities and significant increase in convergence times required to achieve the accuracy of user receiver coordinates of ~ 1 cm. Hence, in this case one cannot take advantage of the integer nature of phase ambiguities because it is compromised by above-mentioned combining procedure.

In the following sections of the paper we formulate an algebraic theory of a user solution employing Integer PPP method to process the measurements of any Global Navigation Satellite System (GNSS) using CDMA (Code-Division Multiple Access) modulation principles. It will be shown that rank deficiency can be eliminated while integer nature of carrier phase ambiguities will still be preserved.

Table. Mathematical models of code and carrier phase measurements at frequencies $f1$, $f2$ of a satellite with the SV number (or a PRN code) of j.

Parameters	Geometric ranges	Wet troposphere delays	User receiver clocks	Code instrument biases in user receivers	Raw carrier phase measurements of user receivers	SV clocks	Code and phase biases in the SV equipment	L1 Ionospheric delay	Integer ambiguities	Errors in measurements
$\rho_{1.i}^{j} =$	R_i^j	$+w_i^j \Delta D_i$	$+dT_i$	$-b_{\rho 1}$		$-dt_i^j$	$-b_{\rho 1}^{j}$	$+I_{1.i}^{j}$		$+\xi_{\rho 1.i}^{j}$
$\rho_{2.i}^{j} =$	R_i^j	$+w_i^j \Delta D_i$	$+dT_i$	$-b_{\rho 2}$		$-dt_i^j$	$-b_{\rho 2}^{j}$	$+\gamma I_{1.i}^{j}$		$+\xi_{\rho 2.i}^{j}$
$L_{1.i}^{j} =$	R_i^j	$+w_i^j \Delta D_i$	$+dT_i$		$+\lambda 1 \psi_{01}$	$-dt_i^j$	$-\lambda 1 \psi_{01}^{j}$	$-I_{1.i}^{j}$	$-\lambda 1 N1^j$	$+\xi_{L1.i}^{j}$
$L_{2.i}^{j} =$	R_i^j	$+w_i^j \Delta D_i$	$+dT_i$		$+\lambda 2 \psi_{02}$	$-dt_i^j$	$-\lambda 2 \psi_{02}^{j}$	$-\gamma I_{1.i}^{j}$	$-\lambda 2 N2^j$	$+\xi_{L2.i}^{j}$

* The subscript i indicates the epoch of the measurements.

Mathematical models of raw GNSS measurements and of their ionosphere-free combinations within CDMA-based integer PPP

The mathematical models of code and carrier phase measurements used by Float and Integer PPP and based on CDMA GNSS signals are presented in the Table. The code models account for satellite-side (and satellite-specific) biases of GNSS signals which affect user receivers. They also account for satellite clock biases (in what follows just 'clocks') which can differ from broadcast clocks transmitted within the navigation message. Satellite-generated carrier phase biases as well as code biases both have impact on user receivers and must be taken into account for Integer PPP.

In CDMA-based GNSS all the satellites (SVs) emit at the same pair of frequency bands $L1$ and $L2$ [2-4]. Hence code and phase biases of different SVs appear in the Table in the same manner, differing only by *j*, an SV index.

In the Table:

R_i^j is the distance between the phase center of a user antenna with unknown coordinates x, y, z and the phase center of the satellite antenna of the *j*-th SV at the epoch *i*;

$f1$, $f2$ are central frequencies and $\lambda 1 = c/f1$, $\lambda 2 = c/f2$ are the wavelengths of the carrier waves emitted by the *j*-th SV on $L1$ and $L2$ frequency bands;

c = 299792458 m/sec is the speed of light in vacuum;

ΔD_i (meters) is the uncompensated part of the vertical tropospheric delay for the user receiver;

w_i^j is the mapping function for the tropospheric delay of the j-th SV at epoch *i*;

$\gamma = (f1/f2)^2 = (n_1/n_2)^2 = (\lambda 2/\lambda 1)^2$, where n_1 and n_2 are the co-prime numbers (for GPS $n_1 = 77$, $n_2 = 60$).

All other notations can be found in the Table.

The models summarily presented in the Table imply that the measurements are already corrected for the main part of the slant tropospheric delay as well as for tidal effects (solid-state, oceanic, and polar), and also for antenna phase centers offsets of both user receivers and SVs, for relativistic effects, for the general relativity delay in the measurements, for phase windup, for the systematic errors in the code and carrier phase measurements caused by the imprecision of broadcast orbits/clocks. Ample literature (see, for example, [1, 5]) is devoted to the computation of such corrections. Multipath errors of measurements are not studied in this paper. It is assumed that they are mitigated by instrumental methods on the measurement generation level.

To exclude the influence of the ionosphere, so-called ionosphere-free linear combinations of the code and carrier phase measurements are typically formed: $\rho_{ifr.i}^j = \left(n_1^2 \rho_{1.i}^j - n_2^2 \rho_{2.i}^j\right) / \left(n_1^2 - n_2^2\right)$ and $L_{ifr.i}^j = \left(n_1^2 L_{1.i}^j - n_2^2 L_{2.i}^j\right) / \left(n_1^2 - n_2^2\right)$. Such combinations, $\rho_{ifr.i}^j$ and $L_{ifr.i}^j$, are perfectly suitable for further processing by Float PPP method, i.e., without ambiguity resolution. However, iono-free combinations are poorly suited for Integer PPP. The formation of iono-free combinations inevitably involves the transformation of the vector of integer ambiguities $\begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$ of phase measurements $L_{1.i}^j$, $L_{2.i}^j$ described by the formula $\begin{bmatrix} N1_{ifr}^j \\ N2_{ifr}^j \end{bmatrix} = \underset{2\times 2}{\boldsymbol{Tr}} \begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$, where $\begin{bmatrix} N1_{ifr}^j \\ N2_{ifr}^j \end{bmatrix}$ is the vector of transformed integer ambiguities introduced as a result of such ionosphere-excluding transformation. Here, $\underset{2\times 2}{\boldsymbol{Tr}}$ is a unimodular matrix (all elements are integers and the determinant equals ±1). Hence, in the framework of Integer PPP the ionosphere-excluding transformation involves two integer ambiguities $N1_{ifr}^j$ and $N2_{ifr}^j$, whereas the ionosphere-free combination $L_{ifr.i}^j$ of the phase measurements $L_{1.i}^j$ and $L_{2.i}^j$ contains just a single ambiguity N_{ifr}^j. To obtain the second ambiguity for combinations $\rho_{ifr.i}^j$ and $L_{ifr.i}^j$, the third ionosphere-free Melbourne–Wübbena (MW) combination $mw_i^j = \frac{n_1 L_{1.i}^j - n_2 L_{2.i}^j}{\Delta n} - \frac{n_1 \rho_{1.i}^j + n_2 \rho_{2.i}^j}{n_1 + n_2}$ ($\Delta n = n_1 - n_2$) is typically added. The MW combination includes the second ambiguity N_{mw}^j. For the three above-mentioned combinations $\rho_{ifr.i}^j$, $L_{ifr.i}^j$, mw_i^j, the following mathematical models can be used:

$$\begin{aligned} \rho_{ifr.i}^j &= R_i^j + w_i^j \Delta D_i + dT_{\rho.ifr.i} - dt_{\rho.ifr.i}^j + \xi_{\rho.ifr.i}^j \\ L_{ifr.i}^j &= R_i^j + w_i^j \Delta D_i + dT_{L.ifr.i} - dt_{L.ifr.i}^j - \lambda_{ifr} N_{ifr}^j + \xi_{L.ifr.i}^j \\ mw_i^j &= b_{mw} - b_{mw.i}^j - \lambda_{mw} N_{mw}^j + \xi_{mw.i}^j \end{aligned} \quad (1)$$

$$j = \overline{1, J_i}$$

where $dT_{\rho.ifr.i} = \frac{n_1^2 \left(dT_i - b_{\rho 1}\right) - n_2^2 \left(dT_i - b_{\rho 2}\right)}{n_1^2 - n_2^2}$ and

$dT_{L.ifr.i} = \frac{n_1^2(dT_i + \lambda 1\psi_{01}) - n_2^2(dT_i + \lambda 2\psi_{02})}{n_1^2 - n_2^2}$ are receiver-side clock biases of ionosphere-free code and phase respectively.

Parameters $dt^j_{\rho.ifr.i} = \frac{n_1^2(dt_i^j + b^j_{\rho1}) - n_2^2(dt_i^j + b^j_{\rho2})}{n_1^2 - n_2^2}$ and $dt^j_{L.ifr.i} = \frac{n_1^2(dt_i^j + \lambda 1\psi^j_{01}) - n_2^2(dt_i^j + \lambda 2\psi^j_{02})}{n_1^2 - n_2^2}$ are satellite-side clock biases of ionosphere-free code and phase transmitted by the j-th SV in the i-th time epoch. Next, $\lambda_{ifr} = \frac{\lambda 1\lambda 2}{n_1\lambda 2 - n_2\lambda 1} = \frac{n_1}{n_1^2 - n_2^2}\lambda 1 = \frac{n_2}{n_1^2 - n_2^2}\lambda 2$ is the ionosphere-free wavelength (for GPS $\lambda_{ifr} \approx 0.0063 m = 6.3\,mm$). Further on,

$$N^j_{ifr} = n_1 N1^j - n_2 N2^j,\ j = \overline{1, J_i} \qquad (2)$$

is the integer ionosphere-free linear combination of the integer ambiguities $N1^j$, $N2^j$ of phase measurements at the original frequencies $f1$, $f2$, while, $b_{mw} = \frac{n_1}{n_1+n_2}b_{\rho1} + \frac{n_2}{n_1+n_2}b_{\rho2} + \frac{n_1}{\Delta n}\lambda 1\psi_{01} - \frac{n_2}{\Delta n}\lambda 2\psi_{02}$, $b^j_{mw.i} = \frac{n_1}{\Delta n}\lambda 1\psi^j_{01} - \frac{n_2}{\Delta n}\lambda 2\psi^j_{02} - \frac{n_1}{n_1+n_2}b^j_{\rho1} - \frac{n_2}{n_1+n_2}b^j_{\rho2}$ are the biases of the MW combinations in the user receiver and hardware delays of the j-th SV $j = \overline{1, J_i}$; $\lambda_{mw} = \frac{n_1}{\Delta n}\lambda 1 = \frac{n_2}{\Delta n}\lambda 2$ is the wavelength of the MW combination (for GPS $\lambda_{mw} \approx 0.862\ m$), while

$$N^j_{mw} = N1^j - N2^j,\quad j = \overline{1, J_i} \qquad (3)$$

is the MW integer ambiguity. Finally,

$\xi^j_{\rho.ifr.i} = \frac{n_1^2\xi^j_{\rho1.i} - n_2^2\xi^j_{\rho2.i}}{n_1^2 - n_2^2}$, $\xi^j_{L.ifr.i} = \frac{n_1^2\xi^j_{L1.i} - n_2^2\xi^j_{L2.i}}{n_1^2 - n_2^2}$,

$\xi^j_{mw.i} = \frac{n_1}{n_1 - n_2}\xi^j_{L1.i} - \frac{n_2}{n_1 - n_2}\xi^j_{L2.i} - \frac{n_1}{n_1+n_2}\xi^j_{\rho1.k} - \frac{n_2}{n_1+n_2}\xi^j_{\rho2.i}$ are measurement errors of ionosphere-free code, phase, and MW combinations respectively, while J_i is the total number of SVs in tracking.

The model (1) is also called 'decoupled clock model' [6], where user receiver and j-th SV clocks are divided into the code clocks $dT_{\rho.ifr.i}$, $dt^j_{\rho.ifr.i}$, the phase clocks $dT_{L.ifr.i}$, $dt^j_{L.ifr.i}$, and the clock biases b_{mw}, b^j_{mw} of the MW combination.

The transformation of the original ambiguity vector $\begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$ into the vector $\begin{bmatrix} N^j_{ifr} \\ N^j_{mw} \end{bmatrix}$ of ambiguities of ionosphere-free combinations (1) is described by the following formula: $\begin{bmatrix} N^j_{ifr} \\ N^j_{mw} \end{bmatrix} = \begin{bmatrix} n_1 & -n_2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$. The matrix $\begin{bmatrix} n_1 & -n_2 \\ 1 & -1 \end{bmatrix}$ is not unimodular; its determinant equals $-\Delta n \neq \pm 1$. This means that not for all integer variables in the space N^j_{ifr}, N^j_{mw} there exist matching integer variables in the space $N1^j$, $N2^j$. This is a well-known cause of significant increase in the probability of incorrect ambiguity resolution for ionosphere-free combinations $\rho^j_{ifr.i}$, $L^j_{ifr.i}$, mw^j_i with the vector of integer ambiguities $\begin{bmatrix} N^j_{ifr} \\ N^j_{mw} \end{bmatrix}$.

Taking formula (3) into account, the equation (2) can be transformed to the form $N^j_{ifr} = \Delta n N1^j + n_2 N^j_{mw}$. Substituting this value into (1), we obtain the system which includes the vector of integer ambiguities $\begin{bmatrix} N1^j \\ N^j_{mw} \end{bmatrix}$

$$\begin{aligned}
&\rho^j_{ifr.i} = R^j_i + w^j_i\Delta D_i + dT_{\rho.ifr.i} - dt^j_{\rho.ifr.i} + \xi^j_{\rho.ifr.i} \\
&L^j_{ifr.i} = R^j_i + w^j_i\Delta D_i + dT_{L.ifr.i} - dt^j_{L.ifr.i} - \lambda_{\Delta nifr}N1^j - \lambda_{n_2 ifr}N^j_{mw} + \xi^j_{L.ifr.i} \\
&mw^j_i = b_{mw} - b^j_{mw.i} - \lambda_{mw}N^j_{mw} + \xi^j_{mw.i} \\
&j = \overline{1, J_i}
\end{aligned} \qquad (4)$$

where $\lambda_{\Delta nifr} = \Delta n\lambda_{ifr}$ $(for\ GPS \approx 0.107\ m)$,

$\lambda_{n_2 ifr} = n_2\lambda_{ifr}$ $(for\ GPS \approx 0.378\ m)$. The transformation of the vector of original integer ambiguities $\begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$ into the vector $\begin{bmatrix} N1^j \\ N^j_{m} \end{bmatrix}$ of integer ambiguities of ionosphere-free combinations (4) is done with the help of this formula: $\begin{bmatrix} N1^j \\ N^j_{mw} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$. The matrix $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ is unimodular so, its determinant equals –1. Substituting (2) and (3) into (1), we obtain the system with the vector of integer ambiguities $\begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$,

$$\rho^j_{ifr.i} = R^j_i + w^j_i \Delta D_i + dT_{\rho.ifr.i} - dt^j_{\rho.ifr.i} + \xi^j_{\rho.ifr.i}$$
$$L^j_{ifr.i} = R^j_i + w^j_i \Delta D_i + dT_{L.ifr.i} - dt^j_{L.ifr.i} - \lambda_{n_1 ifr} N1^j + \lambda_{n_2 ifr} N2^j + \xi^j_{L.ifr.i}$$
$$mw^j_i = b_{mw} - b^j_{mw.i} - \lambda_{mw} N1^j + \lambda_{mw} N2^j + \xi^j_{mw.i}$$
$$j = \overline{1, J_i} \qquad (5)$$

where $\lambda_{n_1 ifr} = n_1 \lambda_{ifr}$ $\left(for\ GPS \approx 0.484\ m\right)$. We see that the matrix of the unimodular transformation of the original integer ambiguities $\begin{bmatrix} N1^j \\ N2^j \end{bmatrix}$ for the system (5) is singular and, hence, unimodular. Thus, the system (5) regarding the probability of the correct ambiguity resolution of phase measurements is equivalent to system (4). Further on we prefer to use the measurement model in the form (4) for the simplicity sake.

Computing user solutions

Assuming that the clock biases of ionosphere-free code $dt^j_{\rho.ifr.i}$ and phase $dt^j_{L.ifr.i}$ as well as the hardware biases b^j_{mwi} of MW-combination of the j-th SV are known from the network solution, the system of nonlinear equations (4) for ionosphere-free measurements can be presented in the linearized form thus:

$$\Delta\rho^j_{ifr.i} = h^j_x \Delta x + h^j_y \Delta y + h^j_z \Delta z + w^j_i \Delta D_i + dT_{\rho.ifr.i} + \xi^j_{\rho.ifr.i}$$
$$\Delta L^j_{ifr.i} = h^j_x \Delta x + h^j_y \Delta y + h^j_z \Delta z + w^j_i \Delta D_i + dT_{L.ifr.i} -$$
$$- \lambda_{\Delta nifr} N1^j - \lambda_{n_2 ifr} N^j_{mw} + \xi^j_{L.ifr.i}$$
$$\Delta mw^j_i = b_{mw} - \lambda_{mw} N^j_{mw} + \xi^j_{mw.i}$$
$$j = \overline{1, J_i} \qquad (6)$$

where $\Delta\rho^j_{ifr.i} = \rho^j_{ifr.i} - R^j_{c.i} + dt^j_{\rho,ifr.i}$, $\Delta L^j_{ifr.i} = L^j_{ifr.i} - R^j_{c.i} + dt^j_{L,ifr.i}$, $\Delta mw^j_i = mw^j + b^j_{mw.i}$, are the residuals of ionosphere-free combinations of code $\rho^j_{ifr.i}$, of carrier phase measurements $L^j_{ifr.i}$, and of MW combination mw^j_i; $R^j_{c.i}$ is the distance between the coarse position of the user receiver x_c, y_c, z_c (the point of linearization at the i-th epoch) and the phase center of the antenna of the j-th SV; $\Delta x = x - x_c$, $\Delta y = y - y_c$, $\Delta z = z - z_c$ are the corrections to the coarse user receiver coordinates x_c, y_c, z_c; $h^j_{x.i} = \left(x_c - x^j_i\right) / R^j_{c.i}$, $h^j_{y.i} = \left(y_c - y^j_i\right) / R^j_{c.i}$, are directional cosines of a unit vector pointed from the antenna phase center of the j-th SV towards the coarse position of the user receiver (x_c, y_c, z_c).

Further on, all vectors and matrices (as opposed to scalar quantities) will be printed in bold-face type with the indication of their dimensions given underneath; for brevity, the subscript *ifr* will be omitted everywhere. The system of linearized equations (6) can be rewritten in a matrix form:

$$\underset{3J_i\times 1}{\boldsymbol{Y}_i} = \underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i} \cdot \underset{(7+2J_i)\times 1}{\boldsymbol{x}_i} + \underset{3J_i\times 1}{\boldsymbol{\Xi}_i} \qquad (7)$$

where

$$\underset{3J_i\times 1}{\boldsymbol{Y}_i} = \left[\Delta\rho^1_i \ \cdots \ \Delta\rho^{J_i}_i \ \ \Delta L^1_i \ \cdots \ \Delta L^{J_i}_i \ \ \Delta mw^1_i \ \cdots \ \Delta mw^{J_i}_i\right] \qquad (8)$$

$(3J_i\times 1)$ is the vector of ionosphere-free linear combinations of code, carrier phases, and MW combinations;

$$\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i} = \left[\Delta x \ \ \Delta y \ \ \Delta z \ \ \Delta D_i \ \ dT_{\rho.i} \ \ dT_{L.i} \right.$$
$$\left. b_{mw} \ \ N1^1 \ \cdots \ N1^{J_i} \ \ N^1_{mw} \ \cdots \ N^{J_i}_{mw}\right]^T \qquad (9)$$

is the state vector (variables to be estimated); $\underset{3J_i\times 1}{\boldsymbol{\Xi}_i} =$ $= \left[\xi^1_{\rho.i} \ \cdots \ \xi^{J_i}_{\rho.i} \ \ \xi^1_{L.i} \ \cdots \ \xi^{J_i}_{L.i} \ \ \xi^1_{mw.i} \ \cdots \ \xi^{J_i}_{mw.i}\right]$ is the vector of the measurement errors of ionosphere-free combinations of the code $\xi^{j_i}_{\rho.i}$, phase measurements $\xi^{j_i}_{L.i}$ and MW combinations $\xi^{j_k}_{mwk}$ in meters;

$$\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i} = \begin{bmatrix} \underset{J_i\times 4}{\boldsymbol{A}_i} & \underset{J_i\times 1}{\boldsymbol{1}} & \underset{J_i\times 1}{\boldsymbol{0}} & \underset{J_i\times 1}{\boldsymbol{0}} & \underset{J_i\times J_i}{\boldsymbol{0}} & \underset{J_i\times J_i}{\boldsymbol{0}} \\ \underset{J_i\times 4}{\boldsymbol{A}_i} & \underset{J_i\times 1}{\boldsymbol{0}} & \underset{J_i\times 1}{\boldsymbol{1}} & \underset{J_i\times 1}{\boldsymbol{0}} & -\lambda_{\Delta nifr} \underset{J_i\times J_i}{\boldsymbol{E}} & -\lambda_{n_2 ifr} \underset{J_i\times J_i}{\boldsymbol{E}} \\ \underset{J_i\times 4}{\boldsymbol{0}} & \underset{J_i\times 1}{\boldsymbol{0}} & \underset{J_i\times 1}{\boldsymbol{0}} & \underset{J_i\times 1}{\boldsymbol{1}} & \underset{J_i\times J_i}{\boldsymbol{0}} & -\lambda_{mw} \underset{J_i\times J_i}{\boldsymbol{E}} \end{bmatrix} \qquad (10)$$

is the design matrix for the state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9).

The notation of the blocks in matrix (10) is as follows:

$\underset{J_i\times 4}{\boldsymbol{A}_i}$ is the matrix of the form

$$\underset{J_i\times 4}{\boldsymbol{A}_i} = \begin{bmatrix} h^1_{x.i} & h^1_{y.i} & h^1_{z.i} & w^1_i \\ \cdot & \cdot & \cdot & \cdot \\ h^{J_i}_{x.i} & h^{J_i}_{y.i} & h^{J_i}_{z.i} & w^{J_i}_i \end{bmatrix},$$

block $\underset{J_i\times 1}{\boldsymbol{0}}$ is the null column vector of size J_i,

block $\underset{J_i\times 1}{\boldsymbol{1}}$ is a column vector made up of J_i units,

block $\underset{J_i\times J_i}{\boldsymbol{0}}$ is the null matrix,

block $\underset{J_i\times J_i}{\boldsymbol{E}}$ is the unit matrix.

The covariance matrix $\underset{3J_i\times 3J_i}{\boldsymbol{R}_i}$ of the vector of measurement errors $\underset{3J_i\times 1}{\boldsymbol{\Xi}_i}$ of ionosphere-free combinations $\underset{3J_i\times 1}{\boldsymbol{Y}_i}$ (8) can be linked to the variances of raw measurements by the transformation $\underset{3J_i\times 3J_i}{\boldsymbol{R}_i} = \underset{3J_i\times 4J_i}{\boldsymbol{T}}\,\underset{4J_i\times 4J_i}{\boldsymbol{R}_{ini.i}}\,\underset{4J_i\times 3J_i}{\boldsymbol{T}^T}$, where $\underset{4J_i\times 4J_i}{\boldsymbol{R}_{ini.i}}$ is the diagonal covariance matrix of raw measurement errors $\xi^j_{\rho 1.i}$, $\xi^j_{\rho 2.i}$, $\xi^j_{L1.i}$, $\xi^j_{L2.i}$ given in the Table; $\underset{3J_i\times 4J_i}{\boldsymbol{T}}$ is the matrix of the form,

$$\underset{3J_i\times 4J_i}{\boldsymbol{T}} = \begin{bmatrix} \underset{J_i\times J_i}{\boldsymbol{T}_{11}} & \underset{J_i\times J_i}{\boldsymbol{T}_{12}} & \underset{J_i\times J_i}{\mathbf{0}} & \underset{J_i\times J_i}{\mathbf{0}} \\ \underset{J_i\times J_i}{\mathbf{0}} & \underset{J_i\times J_i}{\mathbf{0}} & \underset{J_i\times J_i}{\boldsymbol{T}_{23}} & \underset{J_i\times J_i}{\boldsymbol{T}_{24}} \\ \underset{J_i\times J_i}{\boldsymbol{T}_{31}} & \underset{J_i\times J_i}{\boldsymbol{T}_{32}} & \underset{J_i\times J_i}{\boldsymbol{T}_{33}} & \underset{J_i\times J_i}{\boldsymbol{T}_{34}} \end{bmatrix},$$

where the blocks comprising $\underset{3J_i\times 4J_i}{\boldsymbol{T}}$ are diagonal matrices in the form

$$\underset{J_i\times J_i}{\boldsymbol{T}_{11}} = \underset{J_i\times J_i}{\boldsymbol{T}_{23}} = \begin{bmatrix} n_1^2/(n_1^2-n_2^2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & n_1^2/(n_1^2-n_2^2) \end{bmatrix},$$

$$\underset{J_i\times J_i}{\boldsymbol{T}_{12}} = \underset{J_i\times J_i}{\boldsymbol{T}_{24}} = \begin{bmatrix} -n_2^2/(n_1^2-n_2^2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdot & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n_2^2/(n_1^2-n_2^2) \end{bmatrix},$$

$$\underset{J_i\times J_i}{\boldsymbol{T}_{31}} = \begin{bmatrix} -n_1/(n_1+n_2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n_1/(n_1+n_2) \end{bmatrix},$$

$$\underset{J_i\times J_i}{\boldsymbol{T}_{32}} = \begin{bmatrix} -n_2/(n_1+n_2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n_2/(n_1+n_2) \end{bmatrix},$$

$$\underset{J_i\times J_i}{\boldsymbol{T}_{33}} = \begin{bmatrix} n_1/\Delta n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & n_1/\Delta n \end{bmatrix},\ \underset{J_i\times J_i}{\boldsymbol{T}_{34}} = \begin{bmatrix} -n_2/\Delta n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -n_2/\Delta n \end{bmatrix}.$$

Matrix $\underset{3J_i\times 3J_i}{\boldsymbol{R}_i}$ must be taken into account in ambiguity resolution algorithms inasmuch as ambiguities N_1^j, N_{mw}^j, $j=\overline{1,J_i}$ are included into the state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9).

For the matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10), the corresponding null-space matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}$ (a matrix composed from all the base vectors of its null-space as columns) is as follows:

$$\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i} = \begin{bmatrix} \underset{4\times 1}{\mathbf{0}} & \underset{4\times 1}{\mathbf{0}} \\ 0 & 0 \\ 0 & 1 \\ \dfrac{\lambda_{mw}}{n_2} & \dfrac{\lambda_{mw}}{2\lambda_{n_2 ifr}} \\ -\dfrac{1}{\Delta n}\underset{J_i\times 1}{\mathbf{1}} & \dfrac{1}{2\lambda_{\Delta nifr}}\underset{J_i\times 1}{\mathbf{1}} \\ \dfrac{1}{n_2}\underset{J_i\times 1}{\mathbf{1}} & \dfrac{1}{2\lambda_{n_2 ifr}}\underset{J_i\times 1}{\mathbf{1}} \end{bmatrix} = \begin{bmatrix} \underset{4\times 1}{\mathbf{0}} & \underset{4\times 1}{\mathbf{0}} \\ 0 & 0 \\ 0 & 1 \\ \dfrac{\lambda_{mw}}{n_2} & \dfrac{n_1+n_2}{2n_2} \\ -\dfrac{1}{\Delta n}\underset{J_i\times 1}{\mathbf{1}} & \dfrac{1}{2\lambda_{\Delta nifr}}\underset{J_i\times 1}{\mathbf{1}} \\ \dfrac{1}{n_2}\underset{J_i\times 1}{\mathbf{1}} & \dfrac{1}{2\lambda_{n_2 ifr}}\underset{J_i\times 1}{\mathbf{1}} \end{bmatrix} \quad (11)$$

where $\underset{4\times 1}{\mathbf{0}}$ is a null-vector. The two columns of matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}$ (11) are lineally independent. Hence, the rank of the matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}$ (11) and rank deficiency of the matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10) both equal two and, thus, the rank of the matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10) equals 5+2J_i. Note that the rank deficiency of the matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10) in the user solution does not depend on J_i. Thus, the system of equations (7) is singular, i.e., has an infinite set of solutions lying in the two-dimensional plane of solutions parallel to the null-space $\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}$ (11). However, since the first five elements of base column vectors of the null-space matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}$ (11) are zeros, the plane of solutions is orthogonal to the state subspace generated by the first five variables Δx, Δy, Δz, ΔD_i, $dT_{p.ifr.i}$, having in mind that the full state space is generated by a full set of original state vectors $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9) [7–10]. That is why the first five coordinates of the points lying in the plane of solution are the same for all the points of this plane and, hence, can be unambiguously estimated. Thus, the system (7) and the corresponding matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10) are not singular in the proper sense of the word: indeed, the first five variables of the system

(7) from the state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9) can be calculated unambiguously. Further on, we will call such systems and the corresponding matrices *semisingular*.

The estimation of unambiguously evaluated states of semisingular systems is based on the theory of S-transformation [11, 12]. In this theory, S-subspace is introduced in the state space (i.e., in the space of all estimated variables). Its dimension equals the rank $5+2J_i$ of the matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10) of the system of linearized equations (7). This subspace is defined by the system of normal equations

$$\underset{2\times(7+2J_i)}{\left(\boldsymbol{S}_i^{\perp}\right)^T}\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i} = \underset{2\times 1}{\boldsymbol{0}}, \tag{12}$$

where $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ is the matrix of the rank two, its column vectors being orthogonal to the Ssubspace. All the points of the state space are projected along the null-space $\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}$ (11) onto the S-subspace. The vector of coordinates of projections forms a transformed state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$. The relationship between the vectors $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ and $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9) is given by the formula $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}} = \underset{(7+2J_i)\times(7+2J_i)}{\boldsymbol{P}_k}\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ [11, 12], where

$$\underset{(7+2J_i)\times(7+2J_i)}{\boldsymbol{P}_i} = \underset{(7+2J_i)\times(7+2J_i)}{\boldsymbol{E}_i} - \underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}\left(\underset{2\times(7+2J_i)}{\left(\boldsymbol{S}_i^{\perp}\right)^T}\underset{(7+2J_i)\times 2}{\boldsymbol{V}_i}\right)^{-1}\underset{2\times(7+2J_i)}{\left(\boldsymbol{S}_i^{\perp}\right)^T}$$

is the projection matrix, and $\underset{(7+2J_i)\times(7+2J_i)}{\boldsymbol{E}_i}$ is a square unit matrix of dimension $(7+2J_i)\times(7+2J_i)$. The first five elements of the transformed state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ are thus unambiguously estimated variables Δx, Δy, Δz, ΔD_i, $dT_{\text{p.ifr.}i}$. The other elements of the transformed state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ are linear combinations of the elements of the original state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9) not included into the first five, which will further be called 'biased estimated variables'. The coefficients of biased estimated variables in linear combinations are determined by the form of the matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$.

In a general case, the matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ can be chosen randomly among the matrices of the rank two. However, in our particular case, the choice of the matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ is constrained by the need to preserve the integer nature of biased estimated variables $N_1^1,\ldots,N_1^{J_i}$, $N_{mw}^1,\ldots,N_{mw}^{J_i}$ included into the original state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9). It can be proven that only one matrix meets such a requirement:

$$\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}} = \begin{bmatrix} \underset{4\times 1}{\boldsymbol{0}} & \underset{4\times 1}{\boldsymbol{0}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \underset{J_i\times 1}{\boldsymbol{1}^r} & \underset{J_i\times 1}{\boldsymbol{0}} \\ \underset{J_i\times 1}{\boldsymbol{0}} & \underset{J_i\times 1}{\boldsymbol{1}^r} \end{bmatrix} \tag{13}$$

where $\underset{J_i\times 1}{\boldsymbol{1}^r}$ is a null-vector, in which the r-th element is replaced by a unit. The value r is the index of the so-called reference SV, which is usually chosen to be the closest to the zenith. The matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ of the form (13) generates the following projection matrix $\underset{(7+2J_i)\times(7+2J_i)}{\boldsymbol{P}_i}$ and the state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$.

$$\underset{(7+2J_i)\times(7+2J_i)}{\boldsymbol{P}_i} =$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\lambda_{\Delta nifr} & 0 & 0 & 0 & 0 & -\lambda_{n_2ifr} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{mw} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix},$$

$$\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta D_i \\ dT_{\rho.i} \\ dT_{L.i} - \lambda_{\Delta nifr} N_1^4 - \dfrac{n_2}{n_1+n_2}\lambda_{mw} N_{mw}^4 \\ b_{mw} - \lambda_{mw} N_{mw}^4 \\ N1^1 - N1^4 \\ N1^2 - N1^4 \\ N1^3 - N1^4 \\ 0 \\ N1^5 - N1^4 \\ N_{mw}^1 - N_{mw}^4 \\ N_{mw}^2 - N_{mw}^4 \\ N_{mw}^3 - N_{mw}^4 \\ 0 \\ N_{mw}^5 - N_{mw}^4 \end{bmatrix} \tag{14}$$

Here, just for the sake of an example, the matrix $\underset{(7+2J_i)\times(7+2J_i)}{\boldsymbol{P}_i}$ and the state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ are shown for a number of SVs J_i=5 while the reference SV is the one with index r=4). We see that when the matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ (13) is used, the integer combinations in the form $N_1^j - N_1^4$, $N_{mw}^j - N_{mw}^4$ $j=\overline{1.5}$ are created. Hence, the integer nature of original ambiguities is preserved.

The estimated values $\underset{(7+2J_i)\times 1}{\hat{\boldsymbol{x}}_{s.i}}$ of the transformed state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ (14) are obtained by solving the expanded system of linear equations which combines systems (7) and (12)

$$\begin{bmatrix} \underset{3J_i\times 1}{\boldsymbol{Y}_i} \\ \underset{2\times 1}{\boldsymbol{0}} \end{bmatrix} = \begin{bmatrix} \underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i} \\ \underset{2\times(7+2J_i)}{\left(\boldsymbol{S}_i^{\perp}\right)^T} \end{bmatrix} \cdot \underset{(7+2J_i)\times 1}{\boldsymbol{x}_i} + \begin{bmatrix} \underset{3J_i\times 1}{\boldsymbol{\Xi}_i} \\ \underset{2\times 1}{\boldsymbol{0}} \end{bmatrix} \tag{15}$$

As it follows from equation $\underset{2\times(7+2J_i)}{\left(\boldsymbol{S}_i^{\perp}\right)^T} \underset{(7+2J_i)\times 1}{\boldsymbol{x}_i} = \underset{2\times 1}{\boldsymbol{0}}$, the elements of the state estimate vector $\underset{(7+2J_i)\times 1}{\hat{\boldsymbol{x}}_{s.i}}$ of the system (15) which stand in places matching the positions of units in column vectors of the matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ (13) must be zeros. This explains the presence of null values in the state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ (14) in places determined by the position of units in column vectors of the matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ (13). Nevertheless, if it is known a priori that a pair of elements of the state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ of system (15) includes two zeros, the calculation of other elements of the vector of solutions of (15) can be accomplished by solving an easier system of linear equations as follows:

$$\underset{3J_i\times 1}{\boldsymbol{Y}_i} = \underset{3J_i\times(5+2J_i)}{\boldsymbol{H}_{cmpr.i}} \underset{(5+2J_i)\times 1}{\boldsymbol{x}_{cmpr.i}} + \underset{3J_i\times 1}{\hat{\mathbf{I}}_i} \tag{16}$$

where $\underset{3J_i\times(5+2J_i)}{\boldsymbol{H}_{cmpr.i}}$ is the compressed matrix obtained from the original $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10), where the two columns corresponding to state variables N_1^r, N_{mw}^r in the original state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9) are omitted; $\underset{(5+2J_i)\times 1}{\boldsymbol{x}_{cmpr.i}}$ is the compressed state vector obtained from the original vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ (14), where zeros are excluded. Hence, positions of units in the two columns of matrix $\underset{(7+2J_i)\times 2}{\boldsymbol{S}_i^{\perp}}$ (13) determine the indices of the two omitted columns in the original design matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10). The explicit inclusion of these units into the layout of the S-matrix not only helps to preserve the integer nature of biased estimated variables N_1^1, … $N_1^{J_i}$,N_{mw}^1…$N_{mw}^{J_i}$, included in the original state vector $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_i}$ (9), but also makes the rank of the compressed design matrix $\underset{3J_i\times(5+2J_i)}{\boldsymbol{H}_{cmpr.i}}$ to match the rank 5+2J_i of the original design matrix $\underset{3J_i\times(7+2J_i)}{\boldsymbol{H}_i}$ (10). Thus, the rank of the compressed design matrix $\underset{3J_i\times(5+2J_i)}{\boldsymbol{H}_{cmpr.i}}$ in the system (16) becomes equal to the dimension of the compressed state vector $\underset{(5+2J_i)\times 1}{\boldsymbol{x}_{cmpr.i}}$. The unambiguous solution of the system (16) can be obtained if the number of rows in $\underset{3J_i\times(5+2J_i)}{\boldsymbol{H}_{cmpr.i}}$ is greater or equal to the dimension of the corresponding state vector $\underset{(5+2J_i)\times 1}{\boldsymbol{x}_{cmpr.i}}$, i.e., if the condition 3J_i ≥ 5+2J_i holds. Hence, the number of SVs required for a successful solution of the user navigation problem is $J_i \geq 5$. It is easy to see that upon solving the system of linear equations (16), the state estimates $\underset{(7+2J_i)\times 1}{\boldsymbol{x}_{s.i}}$ (14), excluding those which are zeros, will be obtained.

The vector of ionosphere-free combinations of observables $\underset{3J_i\times 1}{\boldsymbol{Y}_i}$ (8), matrix $\underset{3J_i\times(5+2J_i)}{\boldsymbol{H}_{cmpr.i}}$ as well as the covariance matrix $\underset{3J_i\times 3J_i}{\boldsymbol{R}_i}$ of the measurement noise $\underset{3J_i\times 1}{\Xi_i}$ of $\underset{3J_i\times 1}{\boldsymbol{Y}_i}$ (8) are included in the system of linearized equations (16) and can be calculated for the each i-th time epoch. Based on this, it is possible to process appropriate ambiguity elements of state vector $\underset{(5+2J_i)\times 1}{\boldsymbol{x}_{cmpr.i}}$ in order to resolve integer linear combinations of integer biased estimated variables $N_1^1 \ldots N_1^{J_i}$, $N_{mw}^1 \ldots N_{mw}^{J_i}$.

Unfortunately, due to the size limit of this paper, Kalman filtration algorithms will not be discussed here. We can recommend our readers relevant literature on the methods of linear estimation [13, 14] and ambiguity resolution of phase measurements [15, 16].

Results of processing

We computed user solutions for two versions of the network solution. The first solution was obtained with the network of five European stations. The data set was chosen such that the SV constellation was not changed during the processing. In fact, all the stations of this network received measurements from the same set of six SVs. (Processing of networks with dynamic constellations is more complex and is a matter of future research.) In this particular case it was possible to evaluate PPP convergence time in the float ambiguity mode as well as phase ambiguity resolution. Fig. 1 shows the comparison of three-dimensional positional errors for both Float and Integer PPP. For Float PPP method precise orbit and clock products available from the IGS service (so-called IGS-products) were used. For Integer PPP method we used the same IGS orbits as well as decoupled clocks (code, phase, and Melbourne–Wübbena) calculated by us in the network solution including five stations. The 6-th station which did not take part in this network solution played the role of a user receiver. Fig. 1 shows that for this data set, centimeter-level accuracy of positioning with Integer PPP method was reached in 5 minutes.

The second solution included ten stations of the SDCM network (System for Differential Corrections and Monitoring, green circles in Fig. 2) and the SV constellation was changing during the period of processing. Different stations of the SDCM network use nonidentical receivers; this degrades the accuracy of estimated decoupled clocks and, hence, entails considerable reduction in the accuracy of the user solution.

At present, the quality of decoupled satellite clocks calculated for the SDCM network is unstable. Convergence times significantly vary between stations and for different processing intervals. For this reason, the quality of positioning for the second network is assessed through averaging. Fig. 3 shows the comparison of averaged total positional errors for the Float PPP and Integer PPP methods.

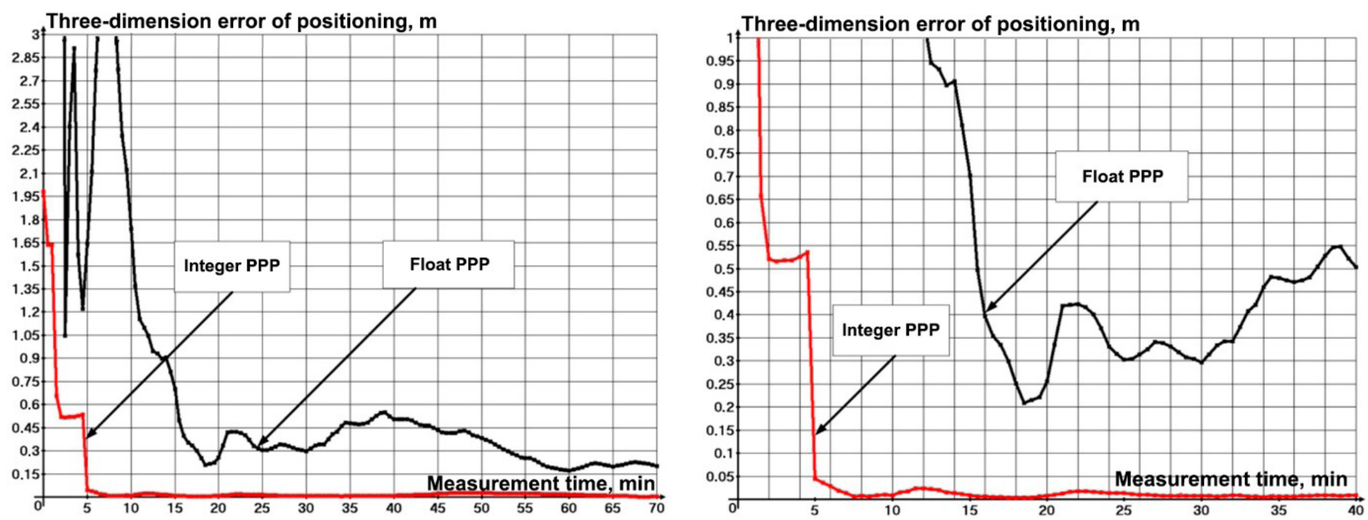

Fig.1. Total (3D) positional errors of Float PPP and Integer PPP methods. The right plot is zoomed

Fig. 2. The stations of the SDCM network employed in the second network solution.

In the second network example (as well as with the first one) precise IGS orbits and clocks were used for Float PPP solution. In the same token, with the Integer PPP method we used same IGS orbit products as well as decoupled clocks (code, phase, and Melbourne–Wubbena) calculated within our own network solution. To compute a user solution, the measurements of two SDCM stations which did not take part in the network solution were used. The scope of averaging included statistics for two stations that acted as user receivers as well five 40-minute processing intervals. As shown in Fig. 3, with the changing constellation of SVs, it takes longer to achieve 5-cm positional accuracy (25–30 min). However, the Integer PPP still offer significant improvement in both convergence and accuracy of positioning compared to the Float PPP. This improvement apparently owes to the use of phase ambiguity resolution.

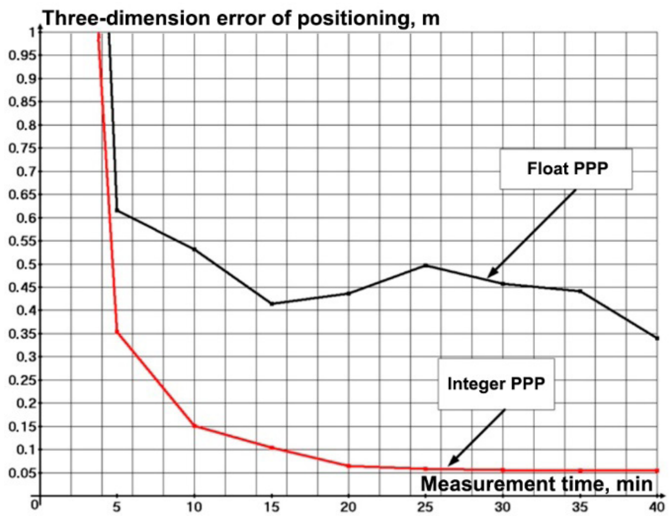

Fig. 3. Averaged total (3D) positional errors for Float and Integer PPP methods

Conclusion

The algebraic principles of the PPP user solution with ambiguity resolution based on the CDMA GNSS signals are studied.

The results of GPS Integer PPP processing demonstrate considerable reduction in convergence times required to achieve high-precision positioning as compared to the Float PPP method.

Our experience indicates urgent necessity to develop fast calibration methods to synchronize network receivers with each other as well as with user receivers in order to achieve better matching of biases of calibrated measurements for those mathematical models that make up the theoretical foundation on which PPP processing algorithms are built.

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